Review: Inverses, Exponentials, and Logarithms - 9/23/16

1 One-to-one Functions

Definition 1.0.1 *A function f is called* ***one-to-one*** *if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.*

Definition 1.0.2 ***Horizontal Line Test****: A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Example 1.0.3 *$f(x) = x^3$ is one-to-one (try the horizontal line test on the graph), but $g(x) = x^2$ is not.*

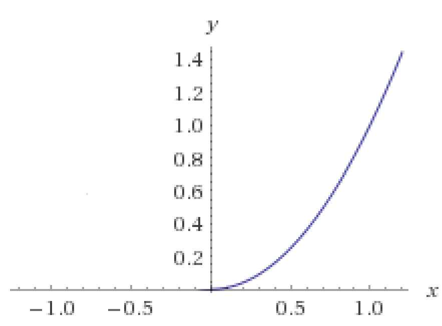
2 Inverse Functions

Definition 2.0.4 *Let f be a one-to-one function with domain A and range B. Then its* ***inverse function*** *f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any y in B.*

When finding an inverse function algebraically:

1. Write $f(x) = y$.
2. Solve for x in terms of y.
3. Swap the x and y.
4. Now $f^{-1}(x) = y$.

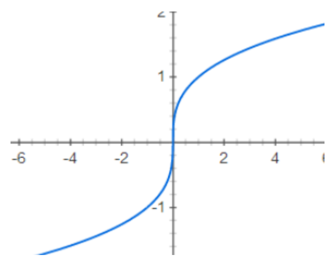
Example 2.0.5 *$f(x) = \sqrt{x}$. Find $f^{-1}(x)$. First write as $y = \sqrt{x}$, then solve for x, so $x = y^2$. Now swap the variables to get $y = x^2$. Then $f^{-1}(x) = x^2$. What is the domain of this? The domain of the inverse function is the range of $f(x)$, so what was the range of $f(x) = \sqrt{x}$? $[0, \infty)$. So the graph will only include the parts greater than or equal to 0.*



The graph of the inverse of a function is reflected over the line $y = x$.

Practice Problems

1. Find the inverse of $f(x) = x - 2$.
2. Find the inverse of $g(x) = \frac{1}{x-5}$.
3. Find the inverse of $h(x) = \frac{1}{x^3+7}$.
4. Sketch the inverse of the following function:

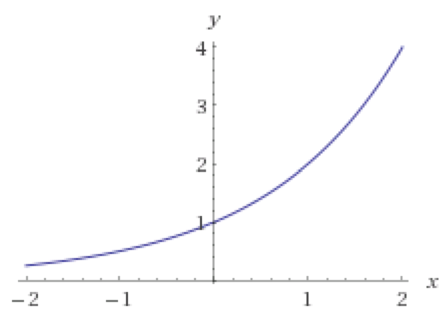


3 Exponential Functions

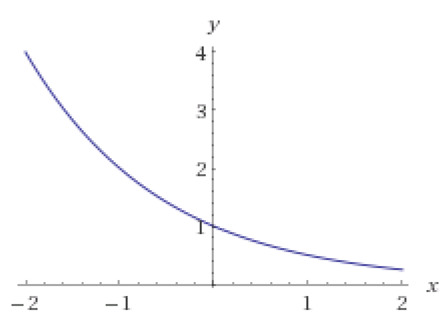
Definition 3.0.6 *An* ***exponential function*** *is a function of the form $f(x) = b^x$ where b is a positive constant.*

The graph of an exponential function depends on the value of b.

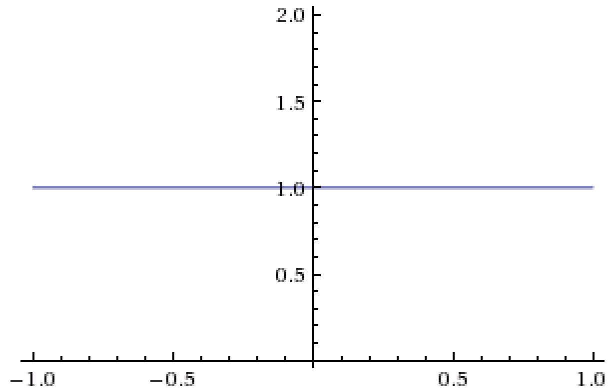
- $b > 1$



- $0 < b < 1$



- $b = 1$



The domain of exponential functions is $(-\infty, \infty)$, and the range is $(0, \infty)$. All exponential functions pass through the point $(0, 1)$.

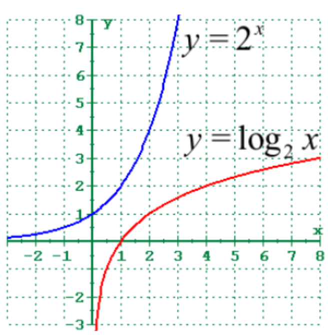
4 Logarithms

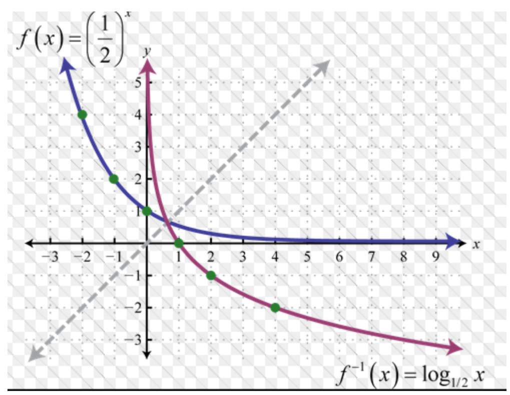
Definition 4.0.7 *A* ***logarithmic function with base*** b *is the inverse of the exponential function* $f(x) = b^x$*. That is,* $\log_b(y) = x \iff b^x = y$*.*

Example 4.0.8 *What is* $\log_2(8)$*? This wants to know: if we have* $2^x = 8$*, what is* x*? So* $\log_2(8) = 3$*.*

Since log is the inverse of exponential functions, they undo each other, so $\log_b(b^x) = x$ and $b^{\log_b(x)} = x$.

Here are examples of graphs of logarithmic functions:





The domain of logarithmic functions is $(0, \infty)$ and range is $(-\infty, \infty)$.

The **natural logarithm** is the logarithm base e. It is written ln.

These are logarithm rules:

Exponent	Logarithm
$b^x \cdot b^y = b^{x+y}$	$\log_b(xy) = \log_b(x) + \log_b(y)$
$\frac{b^x}{b^y} = b^{x-y}$	$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
$b^0 = 1$	$\log_b(1) = 0$
$(b^x)^y = b^{xy}$	$\log_b(x^y) = y \log_b(x)$
	$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Example 4.0.9 *We can write* $\log_2(8) = \log_2(2) + \log_2(4)$*. This means that the exponent that we raise 2 to get 8 is the same as the exponent we raise 2 to get 2 plus the exponent we raise 2 to get 4, i.e.* $3 = 1 + 2$.

Example 4.0.10 $\log_3(x) + \log_9(x) = 6$*. Solve for* x*. Raise everything to a base of 9. Then we have* $9^{\log_3(x)+\log_9(x)} = 9^6$*. Then we have* $(3^2)^{\log_3(x)} \cdot 9^{\log_9(x)} = 9^6$*, so* $3^{2\log_3(x)} \cdot x = 9^6$*, so* $3^{\log_3(x^2)} \cdot x = 9^6$*, so* $x^2 \cdot x = 9^6$*. Now we take the cube root of both sides to get* $x = 9^{6/3} = 9^2 = 81$.

Another way to think about this: Let $\log_3(x) = a$ *and* $\log_9(x) = b$*. Then our equation is* $a+b = 6$*. Now* $3^a = x$ *and* $9^b = x$*, so* $3^a = 9^b$*. Recall that* $3^2 = 9$*, so we can rewrite* $9^b = (3^2)^b = 3^{2b}$*, so* $3^a = 3^{2b}$*. Thus* $a = 2b$*. Substituting this back into our equations, we get that* $2b + b = 6$*, so* $b = 2$*. Then* $a = 2b = 4$*. But remember, we are trying to solve for* x*. We know that* $9^b = x$*,* $9^2 = x = 81$*. Thus* $x = 81$*. Note that we did not have to solve for* a *in order to finish this problem.*

Practice Problems

1. Expand $\log_3(8x)$ into two pieces. Expand it into three pieces. Try expanding it using division instead.
2. Expand $\log_3(8x^3)$ using exponents.
3. Simplify $\log_2(3) + \log_2(7) - 4\log_2(2)$.
4. What is $\log_5(25)$?
5. Solve for x: $\log_4(x) = 3$.
6. Solve for x: $\log_7(x) = 2$.
7. Solve for x: $\log_3(x) = \log_9(x)$.
8. Solve for all possible values of x: $2\log_3(x) = \log_9(x)$.